United Kingdom
Mathematics Trust

# BRITISH MATHEMATICAL OLYMPIAD

## ROUND 2

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# SOLUTIONS

**1.** For a given positive integer $k$, we call an integer $n$ a *$k$-number* if both of the following conditions are satisfied:

(i) The integer $n$ is the product of two positive integers which differ by $k$.

(ii) The integer $n$ is $k$ less than a square number.

Find all $k$ such that there are infinitely many $k$-numbers.

SOLUTION

Note that $n$ is a $k$-number if and only if the equation

$$n = m^2 - k = r(r + k)$$

has solutions in integers $m, r$ with $k \geq 0$. The right-hand equality can be rewritten as

$$k^2 - 4k = (2r + k)^2 - (2m)^2 ,$$

so $k$-numbers correspond to ways of writing $k^2 - 4k$ as a difference of two squares, $N^2 - M^2$ with $N > r$ and $M$ even (which forces $N$ to have the same parity as $k$). Any non-zero integer can only be written as a difference of two squares in finitely many ways (e.g. since the gaps between adjacent squares grow arbitrarily large), so if $k \neq 4$ then there are only finitely many $k$-numbers.

Conversely, if $k = 4$ then setting $m = k + 2$ for $k \geq 0$ shows that there are infinitely many 4-numbers.

**2.** Find all functions from the positive integers to the positive integers such that for all $x, y$ we have:

$$2yf(f(x^2) + x) = f(x + 1)f(2xy).$$

SOLUTION

First substitute $x = 1$ to see that $f(2y) = ky$ for all positive integers $y$, where $k = \frac{2f(f(1)+1)}{f(2)}$ is a positive integer (since $k = f(2)$). Next substitute $x = 2z$ and $y = 1$ to see that $f(2z+1) = kz+1$ for all positive integers $z$. Then substitute $x = 2z + 1$ and $y = 1$ to find that $k = 2$. So $f(x) = x$ for all integers $x \geq 2$. Using $k = \frac{2f(f(1)+1)}{f(2)}$ we find that $f(1) = 1$, and so $f$ is the identity. This is easily checked to satisfy the functional equation.

**3.** The cards from $n$ decks of 52 playing cards are put into boxes, which can each contain at most 2022 cards. A pile of boxes is said to be *regular* if it contains the same number of copies of each card. Show that there exists some $N$ such that, if $n \geq N$, then the boxes can be split into two nonempty regular piles.

ALT?

The cards from $n$ identical decks of cards are put into boxes. Each deck contains 50 cards, labelled from 1 to 50. Each box can contain at most 2022 cards. A pile of boxes is said to be *regular* if that pile contains equal numbers of cards with each label. Show that there exists some $N$ such that, if $n \geq N$, then the boxes can be divided into two nonempty regular piles.

### Solution

For this solution we insist that piles are nonempty.

If a pile contains $x$ copies of card 1 and $y$ copies of card $m$, we call $x - y$ its $(1, m)$ *score*.

**Claim** Suppose we have a collection of $n$ piles whose $(1, m)$ scores sum to zero and have maximum absolute value $k$. The collection can be divided into two piles with $(1, m)$ score zero, provided $n > 2^k$.

**Proof** Induction on $k$. If $k = 1$ we must either have a pile of score 0, or one of score 1 and another of score $-1$.

If there is pile with score $k$ and another with score $-k$ these can be combined to form a pile with score zero, as can all the remaining piles.

If there are piles with score $k$, but none with score $-k$, then each with score $k$ can be combined with a pile with negative score. There are sure to be enough piles with negative score, since the scores sum to zero. This leaves at least $n/2$ piles and the maximum absolute score has decreased, so we are done by induction. (The case where some piles have score $-k$ and none have score $k$ is identical.)

**Corollary** If $n > 2^{k+r-2}$ then we can form $r$ piles with $(1, m)$ score zero.

**Proof** Induction on $r$. The case $r = 2$ is the previous claim. If $r > 2$ then we can form two piles using the claim. One of these consists of at least $2^{k+(r-1)-2}$ of the original piles, so we are done by induction.

Now we assume that cards from decks numbered 1 up to $j$ are placed into boxes each of which can hold $k$ cards. We are sure to have more than $n$ nonempty boxes if we use more than $n \times k/j$ decks.

**Claim** For sufficiently large $n$ we can form two regular piles.

**Proof** Induction on $j$. If $j = 1$ there is nothing to prove.

Setting $m = j > 1$ in the corollary shows that if $n > 2^{k+r-2}$ we can form $r$ piles each of which has $(1, j)$ score zero. Now we can ignore all cards of type $j$, so, by taking $r$ to be sufficiently large, we are done by induction.

ALTERNATIVE

Let there be $M$ nonempty boxes, labelled $b_1, \dots, b_M$. For each $m$, let $x_m = (a_{m1}, \dots, a_{m51})$ be a vector with entries $a_{mi} =$ qty. of card $i + 1$in box $m -$ qty. of card $i$ in box $m$. Then for $K \subseteq \{1, \dots, M\}$, $K$ satisfies our condition iff $\sum_{k \in K} x_k = \mathbf{0}$.

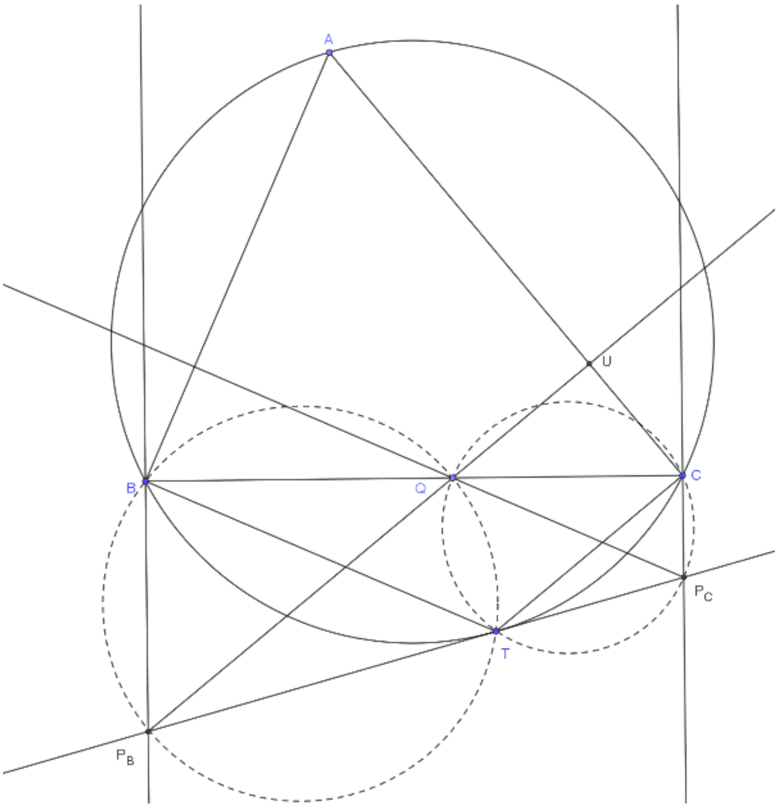
For each $K \subseteq \{1, \dots, M\}$, let $\sum_{k \in K} x_k = y_K$. Thus each $y_K \in [-2019M, 2019M]^{51}$. As there are the same number of each card in the deck, have $y_{\{1,\dots,M\}} = \mathbf{0}$, so $y_K = -y_{K^C}$. We then note that if $y_{K_1} = y_{K_2}$, then $y_{K_1 \Delta K_2^C} = \mathbf{0}$.

We then choose $P$ such that for all $M \geq P$, we have $2^{M-1} > (2 \cdot 2019M)^{51}$. A bit of careful consideration shows this exists. And so by PHP, we can find $K_1 \neq K_2$, for which is not the whole of $\{1, \dots, M\}$, for which $y_{K_1} = y_{K_2}$ as required.

**4.** Let $ABC$ be an acute angled triangle with circumcircle $\Gamma$. Let $l_B$ and $l_C$ be the lines perpendicular to $BC$ which pass through $B$ and $C$ respectively. A point $T$ lies on the minor arc $BC$. The tangent to $\Gamma$ at $T$ meets $l_B$ and $l_C$ at $P_B$ and $P_C$ respectively. The line through $P_B$ perpendicular to $AC$ and the line through $P_C$ perpendicular to $AB$ meet at a point $Q$. Given that $Q$ lies on $BC$, prove that the line $AT$ passes through $Q$.

SOLUTION

Let $P_BQ$ meet $AC$ at $U$.



The right angles in the question show that $CUBP_B$ is cyclic (with diameter $CP_B$) so $\angle UP_BB = \angle C = QP_BB$.

Similarly $\angle CP_CQ = \angle B$.

Now let $T'$ be the second intersection of circles $P_CCQ$ and $P_BBQ$. We claim that $T' = T$.

The right angles show that $T'$ is on the line $P_BP_C$.

The two angle facts established above show that $CT'B = \angle B + \angle C$, so $T'$ lies on the arc $BC$ of circle $ABC$. This proves the claim.

Our claim shows $QT$ is perpendicular to $P_BP_C$.

Next we show that $P_BQ$ is parallel to $TC$. Indeed, we have $\angle TP_BQ = \angle TBQ = \angle P_CTC$ where the first equality is from the cyclicity of $QBP_BT$ and the second is from the alternate segment theorem.

Thus $TC$ is perpendicular to $AC$ so a final application of the alternate segment theorem shows that $AT$ is perpendicular to $P_BP_C$ and we are done.

### ALTERNATIVE

Note that $Q$ is sufficient information to construct $P_B$ and $P_C$.

Extend $AQ$ to meet $\Gamma$ at $T'$, and let the foot of the perpendicular from $P_B$ to $AC$ be $U$. Then $P_BBUC$ is cyclic, as is $ABT'C$.

Consequently: $\angle QP_BB = \angle UP_BB = \angle ACB = \angle A'B$

so $P_BBQT'$ is also cyclic.

In particular, $\angle AT'P_B = 90^\circ$.

But the same holds for $\angle P_CT'A = 90^\circ$.

So $P_B, T', P_C$ are collinear, and thus $T' = T$.

### ALTERNATIVE

Let $\infty_{\perp \ell}$ denote the point at infinity on the line perpendicular to $\ell$. Applying Pappus' theorem to lines $\overline{BQC}$ and $\overline{\infty_{\perp AC}\infty_{\perp BC}\infty_{\perp AB}}$ gives us that $P_BP_CA'$ are collinear where $A' = B\infty_{\perp AB} \cap C\infty_{\perp AC}$ is the point diametrically opposite $A$ on $\odot ABC$. Thus $T \equiv A'$.

If $\tilde{Q} = AA' \cap BC$ then from $BP_BA'\tilde{Q}$, $CP_CA'\tilde{Q}$ cyclic we get $P_B\tilde{Q} \perp AC$ and $P_C\tilde{Q} \perp AB$ so in fact $Q \equiv \tilde{Q}$ and thus $A, Q, T$ are collinear on the diameter passing through $A$.

### ALTERNATIVE

For completeness, here's the slickest Cartesian solution I could come up with. Let:

$$\begin{aligned} A &= (\cos\alpha, \sin\alpha) \\ B &= (\cos\beta, \sin\beta) \\ C &= (\cos\beta, -\sin\beta) \\ T &= (\cos\theta, \sin\theta) \end{aligned}$$

Using $P_B = (x, \sin\beta)$ and solving $(P_B - T) \cdot T = 0$ we get:

$$\begin{aligned} P_B &= (\sec\theta - \sin\beta\tan\theta, \sin\beta) \\ P_C &= (\sec\theta + \sin\beta\tan\theta, -\sin\beta) \end{aligned}$$

where we find $P_C$ by swapping $\beta \longrightarrow -\beta$. Then solving:

$$(P_B - Q) \cdot (A - C) = 0 = (P_C - Q) \cdot (A - B)$$

we get:

$$Q = (\cos\alpha + \cos\beta + \sec\theta + \sin\alpha\tan\theta, \sin\alpha + \tan\theta(\cos\beta - \cos\alpha))$$

As $Q$ lies on $BC$ we have:

$$\begin{gathered} \cos\alpha + \cos\beta + \sec\theta + \sin\alpha\tan\theta = \cos\beta \\ \Rightarrow -1 = \cos\alpha\cos\theta + \sin\alpha\sin\theta = \cos(\alpha - \theta) \\ \Rightarrow |\alpha - \theta| = \pi \end{gathered}$$

We can WLOG take $\theta = \alpha + \pi$ then we have:

$$\begin{aligned} Q &= (\cos\beta, \cos\beta\tan\alpha) \\ A &= (\cos\alpha, \sin\alpha) \\ T &= (-\cos\alpha, -\sin\alpha) \end{aligned}$$

And now we can see $A, Q, T$ are collinear on the line through the origin with gradient $\tan\alpha$.