

United Kingdom
Mathematics Trust

BRITISH MATHEMATICAL OLYMPIAD

ROUND 2

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SOLUTIONS

These solutions are intended as outlines. In particular, they do not represent the full range of approaches possible, nor the difficulties which lie in finding them.

1. A positive integer n is called *good* if there is a set of divisors of n whose members sum to n and include 1. Prove that every positive integer has a multiple which is good.

SOLUTION

Firstly we show that if $m > 1$ is good, then so is $2m$. This is true since some proper divisors of m , including 1 (and hence not including m itself), sum to m ; if we consider all these together with m , they will all be factors of $2m$ which sum to $2m$.

This means that it suffices to prove the claim for odd numbers. The claim holds for 1 since good numbers exist (such as 6, which is $1 + 2 + 3$, for example).

If $a > 1$ is odd and $n = 2^k a$ for some k , then $a + 2a + 4a + \dots + 2^{k-1}a = (2^k - 1)a = n - a$ which is close to n . This value of n will be good if we can find some other factors of n , including 1, which sum to a .

To do this, we write a as a sum of powers of 2, including 1, by writing a in binary, and then choose k to be large enough for all those powers of 2 to be factors of n . (We may take k to be $\lceil \log_2(a) \rceil$, the smallest integer greater than or equal to the base-2 logarithm of a .) None of these powers of 2 are multiples of a so there is no risk that we are using the same factor of n twice.

ALTERNATIVE

A variation on the proof above can be obtained by instead considering $2^{m-1}(2^m - 1)$. This is good for any $m \geq 3$, since

$$2^{m-1}(2^m - 1) = 1 + 2 + 2^2 + \dots + 2^{m-1} + (2^m - 1) + 2(2^m - 1) + \dots + 2^{m-2}(2^m - 1).$$

It can be shown using the Fermat-Euler theorem that any number n is a factor of a number of this form. Indeed, suppose $n = 2^k a$, with a odd. Then the Fermat-Euler theorem tells us that $2^{c\varphi(a)} - 1$ is a multiple of a for any positive integer c (where $\varphi(a)$ is the number of numbers between 1 and a which are coprime to a). This means that if we choose c such that $k < c\varphi(a)$, then we will have n a factor of $2^{c\varphi(a)-1}(2^{c\varphi(a)} - 1)$

ALTERNATIVE

We show that the numbers $n!$ are all good for all $n \geq 3$, which is enough to solve the problem since n is always a factor of $n!$. We proceed by induction, using $3! = 6 = 3 + 2 + 1$ for the base case. For the inductive step, suppose that

$$n! = 1 + \sum_i a_i$$

where a_i are distinct factors of $n!$ other than 1. Then we have

$$(n+1)! = 1 + n + \sum_i (n+1)a_i,$$

and all these factors are different, so $(n+1)!$ is also good.

2. Eliza has a large collection of $a \times a$ and $b \times b$ tiles where a and b are positive integers. She arranges some of these tiles, without overlaps, to form a square of side length n . Prove that she can cover another square of side length n using only one of her two types of tile.

SOLUTION

Number the rows of the $n \times n$ square from 1 to n .

We say that a cell is *special* if it is in the top row of a $b \times b$ square.

In any row, the number of cells that are in a $b \times b$ square is congruent to n , modulo a . Since each special cell has $b - 1$ cells in the $b - 1$ rows underneath it, we can see by induction that the number of special cells in rows $1, b + 1, 2b + 1, \dots$ is congruent to n modulo a , and the number of special cells in rows with other numbers is congruent to 0.

Now, the cells in $b \times b$ squares in row n are all in squares whose top cells are in row $n - b + 1$, and hence the number of special cells in row $n - b + 1$ is congruent to n (modulo a). Hence either $n \equiv 0 \pmod{a}$, in which case a is a multiple of n , or $n - b + 1$ is of the form $rb + 1$, in which case b is a multiple of n .

ALTERNATIVE

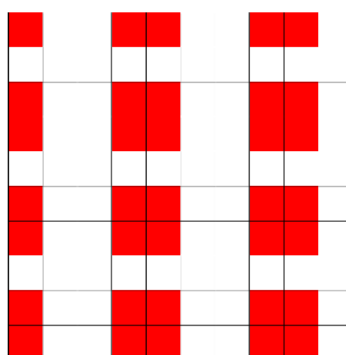
Number the rows of the square to be tiled from 1 to n and suppose that a does not divide n .

If the row number is congruent to 1 modulo a write +1 in every cell in that row, and if the row number is congruent to 0 modulo a write -1 in every cell in that row. Write 0 in every other cell.

The sum of all the entries on the board is exactly n . The entries covered by an $a \times a$ tile sum to 0, and the those covered by a $b \times b$ tile sum to a multiple of b . Thus the total of the entries covered by any collection of tiles is congruent to zero modulo b , so $b|n$ as required.

ALTERNATIVE

Suppose that n is divisible by neither a nor b . Imagine tiling plane with $a \times b$ rectangles and colour in the four corners of each rectangle as shown (for $a = 3, b = 4, n = 10$).



More precisely, we shade the cell with coordinates (x, y) (for $1 \leq x, y \leq n$) just when $a \mid x$ or $x - 1$, and $b \mid y$ or $y - 1$.

It is easy to verify (using the periodicity of the colouring) that each $a \times a$ or $b \times b$ tile covers an even number of shaded cells, but that there are an odd number of shaded cells in total. Thus the $c \times c$ square cannot be tiled with $a \times a$ and $b \times b$ tiles, as desired.

ALTERNATIVE

Stretch the board by $1/a$ in one direction and by $1/b$ in the other. Now each of Eliza's tiles has one side equal to 1. A theorem of De Bruijn now implies that the whole board must have an integer side length, which, after reversing the stretches, implies the result.

REMARK

The second alternative opens up many possible approaches, since De Bruijn's result has many proofs.

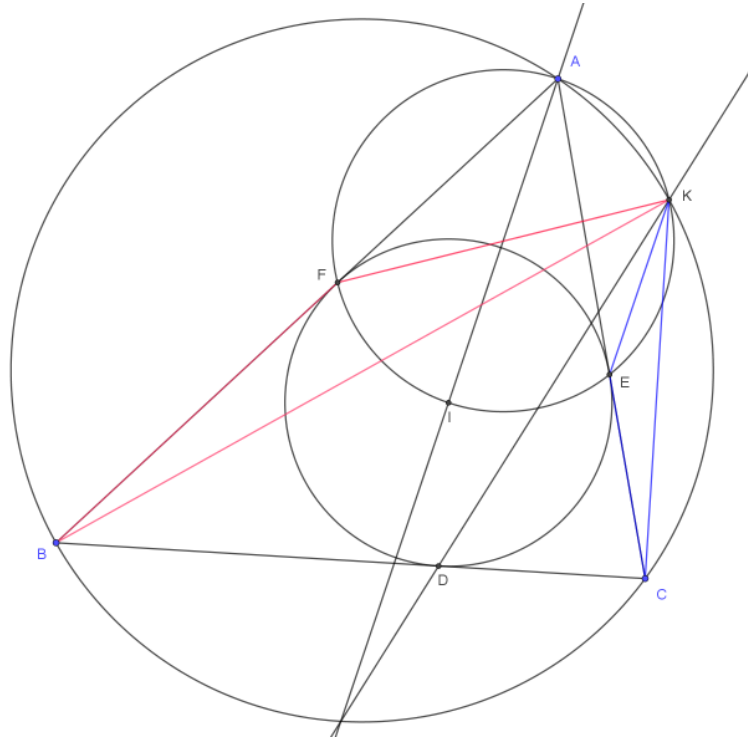
For example we can divide the original board into rectangles whose dimensions are $a/2$ and $b/2$ and apply a standard chessboard colouring. Each of Eliza's tiles covers an equal black and white area, but it can be checked that the board only consists of equal black and white areas if its side is divisible by either a or b .

3. Let ABC be a triangle with $AB > AC$. Its circumcircle is Γ and its incentre is I . Let D be the contact point of the incircle of ABC with BC .

Let K be the point on Γ such that $\angle AKI$ is a right angle.

Prove that AI and KD meet on Γ .

SOLUTION



The line AI meets the circumcircle Γ at the midpoint M of the minor arc BC . We produce MD to cut the circumcircle again at K' . We will show that $\angle AK'I$ is a right-angle and so $K' = K$.

Angle in the same segment gives $\angle MK'B = \angle CBM = \frac{A}{2}$ and so MB is tangent to circle BDK' at B , and the tangent-secant theorem applies so $MB^2 = MD \cdot MK'$. We also know that $MB = MC = MI$, and substituting MI into this relation, the converse of the tangent-secant theorem tells us that MI is tangent to the undrawn circle DIK' at I . Now $\angle DIM + \angle MIB = \angle DIB = 90^\circ - \frac{B}{2}$. Therefore $\angle DIM = \frac{B-C}{2}$ (we have just used the fact that $\angle B < \angle C$). Putting all this together, we find that $\angle AK'I = \angle AK'B - \angle IK'M - \angle MK'B = \angle ACB - \angle MID - \angle MAB = 180^\circ - C - \frac{B-C}{2} - \frac{A}{2} = 180^\circ - \frac{A+B+C}{2} = 90^\circ$ as required.

ALTERNATIVE

We know that AI meets the circumcircle again at M , the midpoint of the arc BC . Since Γ is also the circumcircle of $\triangle KBC$, we know that KD also passes through M precisely if KD is the angle bisector of $\angle BKC$.

Let E, F be the contact points of the incircle with CA, AB , respectively, so $AFIEK$ is cyclic (by the converse of Thales's theorem). Now K , the intersection of Γ and circle $AFIEK$, is the

centre of spiral similarity which carries the directed segment FB to EC . Therefore

$$\frac{KB}{KC} = \frac{BF}{CE} = \frac{BD}{DC}$$

since tangents from a point to a circle have the same length. Now, by the converse of the angle bisector theorem, KD is the angle bisector of angle $\angle CKB$.

ALTERNATIVE

It is possible to adjust the approach above so as to not to require knowledge of the spiral similarity theorem. One way to proceed is by chasing angles to prove that $\triangle KFB$ and $\triangle KEC$ are similar.

4. Matthew writes down a sequence a_1, a_2, a_3, \dots of positive integers. Each a_n is the smallest positive integer, different from all previous terms in the sequence, such that the mean of the terms a_1, a_2, \dots, a_n is an integer. Prove that the sequence defined by $a_i - i$ for $i = 1, 2, 3, \dots$ contains every integer exactly once.

SOLUTION

Write $b_i = a_i - i$. We prove by induction that b_1, \dots, b_n consists of consecutive integers in some order. Let $B_n = \max\{b_1, \dots, b_n\}$. Since $b_1 = 0$, we have $B_n < n$. The mean of the sequence $a_1, a_2, \dots, a_n, n+2+B_n$ is an integer, as it is the sum of the two sequences $1, 2, \dots, n, n+1$ and $b_1, b_2, \dots, b_n, B_n+1$ of consecutive integers, and the mean of any sequence of $n+1$ consecutive integers is an integer (respectively a half-integer) according as n is even (respectively odd). It follows that a_{n+1} is congruent to $n+2+B_n \pmod{n+1}$. Since $n+2+B_n$ is greater than any of a_1, \dots, a_n (as $a_i = i + b_i$ with $i < n+1$ and $b_i < B_n+1$, we know that $a_{n+1} \leq n+2+B_n$, and hence a_{n+1} is equal to $n+2+B_n$ or $1+B_n$ (as $B_n - n < 0$). Thus b_{n+1} is equal to B_n+1 or B_n-n , and the integers b_1, \dots, b_{n+1} are consecutive. This completes the induction. It follows that every integer appears in the sequence b_1, b_2, \dots at most once.

To complete the problem, we need only check that $b_n > 0$ infinitely often and also that $b_n < 0$ infinitely often. For the former, we simply note that if $b_n < 0$, then we cannot also have $b_{n+1} < 0$. Indeed, since $0 = b_1$ is among the consecutive integers b_1, \dots, b_n , the only way we could have $b_{n+1} < 0$ would be if $b_{n+1} = b_n - 1$, in which case $a_{n+1} = a_n$, which is not possible. For the latter, suppose for contradiction that there is some n_0 such that $b_n > 0$ for all $n > n_0$. It follows that for $n > n_0$ we have $b_n = n - \delta$, and hence $a_n = 2n - \delta$, for some δ . Since the mean of $a_1, \dots, a_{n-1}, n - \delta$ is also an integer, it follows that $n - \delta$ must appear in the sequence a_1, \dots, a_{n-1} for all $n > n_0$. If in addition n is odd, we cannot have $n - \delta = a_{n'} = 2n' - \delta$ for an integer $n' > n_0$, and hence $n - \delta$ must even appear in the sequence a_1, \dots, a_{n_0} . But there are infinitely many odd integers $n > n_0$ and only finitely many integers in the range a_1, \dots, a_{n_0} , which is impossible.

ALTERNATIVE

Let τ be the Golden Ratio, so $1/\tau = \tau - 1$. Define $m(n)$ to be the ceiling of n/τ . If $m(n) = m(n-1)$ then define $a(n) = m(n-1)$ (case 1). If $m(n) = m(n-1) + 1$ then define $a(n) = m(n-1) + n$ (case 2). Then $m(n)$ is the mean of $a(1), a(2), \dots, a(n)$ because $a(n) = m(n-1) + kn$ and $m(n) = m(n-1) + k$ always, where k is an integer. Case 1 is $a-1 < (n-1)/\tau < n/\tau < a$, which rearranges to $(a-1)/\tau < n-a < a/\tau$. Case 2 is $(n-1)/\tau < a-n < n/\tau$, which rearranges to $n-1 < (a-1)/\tau < a/\tau < n$. It follows that $a(n)$ is self-inverse, in particular bijective. Note that $a(m(n-1)) < \tau m(n-1) < \tau((n-1)/\tau + 1) < n+1$, so $m(n-1)$ is used by position n . So if we are in case 2, $m(n-1)$ has been used previously. So we are always using the smallest unused $m(n-1) + kn$ (since $k < 0$ is impossible). Hence our sequence is the same as the one in the question. Case 2 has $a(n) - n$ equal to each positive integer once, as n ranges over all positive integers. Case 1 has $a(n) - n$ equal to each negative integer once, as n ranges over all positive integers. Since $a(1) - 1 = 0$, that completes the problem.