United Kingdom
Mathematics Trust

British Mathematical Olympiad

Round 1

Wednesday 16 November 2022

Instructions

1. Time allowed: $3\frac{1}{2}$ hours.
2. Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then write up your best attempt. Do not hand in rough work.
3. One complete solution will gain more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all the problems.
4. Each question carries 10 marks. However, earlier questions tend to be easier. In general you are advised to concentrate on these problems first.
5. The use of rulers, set squares and compasses is allowed, but calculators and protractors are forbidden. *You are strongly encouraged to use geometrical instruments to construct large, accurate diagrams for geometry problems.*
6. Start each question on an official answer sheet on which there is a QR code.
7. If you use additional sheets of (plain or lined) paper for a question, please write the following in the top left-hand corner of each sheet. (i) The question number. (ii) The page number for that question. (iii) The digits following the ':' from the question's answer sheet QR code. Please **do not** write your name or initials on additional sheets.
8. Write on one side of the paper only. Make sure your writing and diagrams are clear and not too faint. (*Your work will be scanned for marking.*)
9. Arrange your answer sheets in question order before they are collected. If you are not submitting work for a particular problem, please remove the associated answer sheet.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 8am GMT on Friday 18 November when the solutions video will be released at https://bmos.ukmt.org.uk
11. **Do not turn over until told to do so.**

Enquiries about the British Mathematical Olympiad should be sent to:

challenges@ukmt.org.uk

www.ukmt.org.uk

1. A road has houses numbered from 1 to n, where n is a three-digit number. Exactly $\frac{1}{k}$ of the numbers start with the digit 2, where k is a positive integer. Find the possible values of n.

2. A sequence of positive integers a_n begins with $a_1 = a$ and $a_2 = b$ for positive integers a and b. Subsequent terms in the sequence satisfy the following two rules for all positive integers n:

$$a_{2n+1} = a_{2n}a_{2n-1}, \quad a_{2n+2} = a_{2n+1} + 4.$$

Exactly m of the numbers $a_1, a_2, a_3, \ldots, a_{2022}$ are square numbers. What is the maximum possible value of m? *Note that m depends on a and b, so the maximum is over all possible choices of a and b.*

3. In an acute, non-isosceles triangle ABC the midpoints of AC and AB are B_1 and C_1 respectively. A point D lies on BC with C between B and D. The point F is such that $\angle AFC$ is a right angle and $\angle DCF = \angle FCA$. The point G is such that $\angle AGB$ is a right angle and $\angle CBG = \angle GBA$. Prove that B_1, C_1, F and G are collinear.

4. Alex and Katy play a game on an 8×8 square grid made of 64 unit cells. They take it in turns to play, with Alex going first. On Alex's turn, he writes 'A' in an empty cell. On Katy's turn, she writes 'K' in two empty cells that share an edge. The game ends when one player cannot move. Katy's score is the number of Ks on the grid at the end of the game. What is the highest score Katy can be sure to get if she plays well, no matter what Alex does?

5. For each integer $n \geq 1$, let $f(n)$ be the number of lists of different positive integers starting with 1 and ending with n, in which each term except the last divides its successor. Prove that for each integer $N \geq 1$ there is an integer $n \geq 1$ such that N divides $f(n)$.

(So $f(1) = 1, f(2) = 1$ and $f(6) = 3$.)

6. A circle Γ has radius 1. A line l is such that the perpendicular distance from l to the centre of Γ is strictly between 0 and 2. A frog chooses a point on Γ whose perpendicular distance from l is less than 1 and sits on that point. It then performs a sequence of jumps. Each jump has length 1 and if a jump starts on Γ it must end on l and vice versa. Prove that after some finite number of jumps the frog returns to a point it has been on before.